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VAGUE CONVERGENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES†

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To the memory of Shlomo Horowitz

ABSTRACT

A sequence (μ_n) of probability measures on the real line is said to converge vaguely to a measure μ if $\int fd\mu_n \to \int fd\mu$ for every continuous function f with *compact* support. In this paper one investigates problems analogous to the classical central limit problem under vague convergence. Let $\|\mu\|$ denote the total mass of μ and δ_0 denote the probability measure concentrated in the origin. For the theory of infinitesimal triangular arrays it is true in the present context, as it is in the classical one, that all obtainable limit laws are limits of sequences of infinitely divisible probability laws. However, unlike the classical situation, the class of infinitely divisible laws is not closed under vague convergence. It is shown that for every probability measure μ there is a closed interval $[0,\lambda]$, $[0,e^{-1}]\subset[0,\lambda]\subset[0,1]$, such that $\beta\mu$ is attainable as a limit of infinitely divisible probability laws iff $\beta\in[0,\lambda]$. In the independent identically distributed case, it is shown that if $(x_1+\cdots+x_n)/a_n$, $a_n\to\infty$, converges vaguely to μ with $0<\|\mu\|<1$, then $\mu=\|\mu\|\delta_0$. If furthermore the ratios a_{n+1}/a_n are bounded above and below by positive numbers, then $L(x)=P[|X_1|>x]$ is a slowly varying function of x. Conversely, if $L(x)$ is slowly varying, then for every $\beta\in(0,1)$ one can choose $a_n\to\infty$ so that the limit measure $=\beta\delta_0$.

0. Introduction

Let μ_n, $n\geqq 1$ and μ be nonnegative finite measures on the Borel sets of the real line. We say that $\mu_n \xrightarrow{c} \mu$ (μ_n *converges to* μ *completely*) if for every bounded continuous function f on the line

$$\int fd\mu_n \to \int fd\mu. \tag{0.1}$$

If (0.1) is required to hold only for $f\in C_0$, where C_0 is the class of continuous functions vanishing at ∞, then we say μ_n *converges to* μ *vaguely* and simply write

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$\mu_n \to \mu$. If X is a real-valued random variable on some probability space, we denote by $\mathscr{L}(X)$ its probability distribution function (p.d.f.). The classical central limit problem concerns the behavior of the p.d.f.s of normalized sums of independent random variables, or of row sums of triangular arrays with independent random variables in each row, under complete convergence. Here we investigate analogous problems for vague convergence.

In the context of sums of independent random variables vague convergence is more difficult, and arguably less natural, than complete convergence. The basic reason is that if $F_n \xrightarrow{c} F$ and $G_n \xrightarrow{c} G$, then $F_n * G_n \xrightarrow{c} F * G$, but this is no longer true under vague convergence. (Take for example F_n to have unit mass at n and G_n to have unit mass at $-n$). Though our prime concern is with developing the appropriate theory for vague convergence, some of our results give new insight and approach to the classical, complete convergence theory as well.

Our main conclusions are as follows. For the theory of infinitesimal triangular arrays it is true in the vague convergence context, as it is in the classical one, that all obtainable limit laws are limits of sequences of infinitely divisible p.d.f.s. However, unlike the complete convergence case, the class of infinitely divisible p.d.f.s is *not* closed under vague convergence. Indeed we show that for every p.d.f. F there exists a nonempty closed interval K_F such that βF is attainable as a limit of infinitely divisible p.d.f.s if $\beta \in K_F$. We know that $[0, e^{-1}] \subset K_F \subset [0, \lambda]$; it would be interesting to know the right end point of the interval K_F.

On the other hand, when dealing with normalized sums of independent random variables with a common p.d.f. F, the situation is quite otherwise: only for exceptional F can anything more be obtained by vague convergence than by complete convergence. Indeed we show that if S_n is the normalized n-th partial sum, $S_n = (X_1 + \cdots + X_n)/a_n$, with $a_n \to \infty$, then $\mathscr{L}(S_n)$ can converge vaguely to a nonzero limit without converging completely only if the limit distribution is concentrated in the origin. If furthermore the ratios a_{n+1}/a_n are bounded uniformly above and below by positive numbers, then

$$L(x) = 1 - F(x) + F(-x-)$$

must be slowly varying. Conversely if L is slowly varying, for every $\beta \in (0, 1)$ one can choose a_k so that the limit law has mass β at the origin and no other mass on the line.

In our discussion of triangular arrays, we develop a notion of centering introduced by Feller in [2]. As remarked by Feller, this can result in considerable convenience in the study of convergence questions (either complete or vague). The details were not pursued by Feller, and turn out to be surprisingly sensitive.

Before proceeding we introduce some notation which will be used subsequently.

If F is a p.d.f. we will denote the corresponding probability measure by F itself. G is a *subprobability distribution function* (s.p.d.f.) if $G = \beta F$, where β is some real number, $0 \leqq \beta \leqq 1$, and F is a p.d.f. The total variation of G, denoted by $\|G\|$, equals β in this case. If $\beta = 0$ we will simply write $G = 0$.

δ_x will denote the p.d.f. with unit mass at x. F^{*k} denotes the k-fold convolution of the s.p.d.f. F, $F^{*0} = \|F\| \delta_0$.

For a real-valued random variable X its expectation and variance are denoted by $\mathrm{E}X$ and $\mathrm{Var}(X)$. $N(\mu, \sigma^2)$ denotes the Gaussian p.d.f. with mean μ and variance σ^2. ch.f. is short for "characteristic function".

If X is a random variable, its symmetrization ${}^\circ X$ means $X - \tilde{X}$, where X and $\tilde{X}$ are independent and identically distributed. If F is a s.p.d.f., its symmetrization will be denoted by ${}^\circ F$, which means $F * G$, where $G((a, b)) = F((-b, -a))$ for every interval (a, b).

A sequence of nonnegative measures (μ_n) will be called *tight* if $\sup_n \|\mu_n\| < \infty$ and for any $\varepsilon > 0$ there exists $A > 0$ such that $\sup_n \mu_n(\{x : |x| > A\}) < \varepsilon$.

1. A concentration function inequality

If F is a p.d.f. on the line, its concentration function Q_F is defined by

$$Q_F(y) = \begin{cases} 0, & y \leqq 0, \\ \sup_x F(x + y/2) - F(x - y/2 -), & y > 0. \end{cases} \tag{1.1}$$

Q_F is a (right-continuous) p.d.f.

Let $X_1, X_2, \cdots, X_n$ be independent r.v.s and ${}^\circ X_1, {}^\circ X_2, \cdots, {}^\circ X_n$ the corresponding symmetrized r.v.s. Let S_n and ${}^\circ S_n$ denote their respective sums. The following concentration function inequality is well known, see [4], theorem 2.2.4, and will be used later:

$$Q_{S_n}(a) \leqq a_0 a \left\{ \sum_{k=1}^{n} \left(\int_{|x|<a_k} x^2 d^\circ F_k + a_k^2 \int_{|x|\geqq a_k} d^\circ F_k \right) \right\}^{-1/2}, \tag{1.2}$$

where a_0 is an absolute constant and $a_1, \cdots, a_k$ are positive constants less than or equal to a.

We would like to note two simple consequences of this inequality in the form of Propositions 1.1 and 1.2.

PROPOSITION 1.1. *Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed, r.v.s and let S_n denote the n-th partial sum. If for some $\lambda > 0$*

$$\limsup_n \mathrm{P}\{|n^{-1/2}S_n| < \lambda\} > 0 \tag{1.3}$$

then $\mathrm{E}X_1^2 < \infty$ *and* $\mathrm{E}X_1 = 0$.

PROOF. It is enough to prove $\mathrm{E}X_1^2 < \infty$ because then $\mathrm{E}X_1 = 0$ follows from (1.3) and Kolmogorov's strong law of large numbers. We now apply (1.2) to $n^{-1/2}X_k$, $1 \leqq k \leqq n$, to get for $a = a_1 = \cdots = a_n$, $n \geqq 1$

$$Q_{n^{-1/2}S_n}(a) \leqq a_0 a \left(\int_{|x|<an^{1/2}} x^2 d^\circ F \right)^{-1/2} \tag{1.4}$$

where F is the p.d.f. of X_1. If $\mathrm{E}X_1^2 = \infty$, then $\mathrm{E}^\circ X_1^2 = \infty$ and in (1.4) we have the left side tending to 0 for every $a > 0$ as $n \to \infty$. This contradicts (1.3), and the proposition is proved.

The conclusion of the proposition is well known to be equivalent to $n^{-1/2}S_n$ converging completely to the standard Gaussian distribution. Let $\{X_k\}$ and $\{S_n\}$ be as in Proposition 1.1. Let

$$A_n = \max_{1 \leqq k \leqq n} |S_k|, \qquad \varphi(n) = (2n/\log\log n)^{1/2}.$$

Jain and Pruitt showed [5] that $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 < \infty$ is sufficient for

$$\liminf_n \frac{A_n}{\varphi(n)} = \pi/8^{1/2} \qquad \text{a.s.} \tag{1.5}$$

Using Proposition 1.1 we can show

PROPOSITION 1.2. *If* $\liminf_n \varphi(n)^{-1}A_n < \infty$ *a.s. then* $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 < \infty$.

After we announced this result [6] it came to our attention that Csáki [1] also observed it.

PROOF. Let $\psi(n) = [\varphi(n)^2]$, where $[x]$ is the greatest integer $\leqq x$. Assume the hypothesis of the theorem. Then by the 0–1 law there exists a finite constant c_0 such that a.s.

$$\liminf_n \frac{A_n}{\varphi(n)} = c_0.$$

Introduce the following notation:

$$n' = [n/\log\log n][\log\log n],$$

$$S^{(j)}_{\psi(n)} = X_{(j-1)\psi(n)+1} + X_{(j-1)\psi(n)+2} + \cdots + X_{j\psi(n)}, \quad 1 \leqq j \leqq [\log\log n],$$

$$U_j = S^{(1)}_{\psi(n)} + \cdots + S^{(j)}_{\psi(n)}, \qquad U_0 = 0.$$

Then for $c > 0$,

$$\begin{aligned}
\mathrm{P}\left[\frac{A_n}{\sqrt{\psi(n)}} < c\right] &\leqq \mathrm{P}\left[\frac{A_{n'}}{\sqrt{\psi(n)}} < c\right] \\
&\leqq \mathrm{P}\left[\max_{1\leqq j\leqq[\log\log n]} \frac{U_j}{\sqrt{\psi(n)}} < c\right] \\
&\leqq \mathrm{P}\left[\max \frac{|U_j| + |U_{j-1}|}{\sqrt{\psi(n)}} < 2c\right] \\
&\leqq \mathrm{P}\left[\max \frac{|U_j - U_{j-1}|}{\sqrt{\psi(n)}} < 2c\right] \\
&= \mathrm{P}\left[\max \frac{|S^{(j)}_{\psi(n)}|}{\sqrt{\psi(n)}} < 2c\right] \\
&= \left(\mathrm{P}\left[\frac{|S^{(j)}_{\psi(n)}|}{\sqrt{\psi(n)}} < 2c\right]\right)^{[\log\log n]}
\end{aligned}$$

To obtain a contradiction, assume the conclusion of the proposition does not hold. By Proposition 1.1

$$\lim_n \mathrm{P}\left[\frac{|S^{(1)}_{\psi(n)}|}{\sqrt{\psi(n)}} < 2c\right] = 0$$

and so the above string of inequalities yields

$$\mathrm{P}\left[\frac{A_n}{\sqrt{\psi(n)}} < c\right] = O\left(\frac{1}{\log^2 n}\right).$$

By the Borel–Cantelli lemma, $A_{2^j}/\sqrt{\psi(2^j)} < c$ holds for only finitely many j, a.s. So for n large, $2^j < n \leqq 2^{j+1}$, one has

$$A_n \geqq A_{2^j} \geqq c\sqrt{\psi(2^j)} \geqq \frac{c}{2}\sqrt{\psi(2^{j+1})} \geqq \frac{c}{2}\sqrt{\psi(n)}.$$

Choosing $c > 2c_0$ gives the desired contradiction.

2. Vague convergence and characteristic functions

In the study of complete convergence the characteristic function (ch.f.) is a very useful tool. In part this is because of the availability of the Lévy continuity theorem, for which we know of no substitute in the study of vague convergence. However, the following holds. Let C_K = class of continuous functions with compact support.

PROPOSITION 2.1. *Let (F_n) and (G_n) be sequences of s.p.d.f.s and let f_n and g_n be the ch.f.s of F_n and G_n respectively. The following two conditions are equivalent:*

$$\lim_n \int hd(F_n - G_n) = 0 \qquad \text{for all } h \in C_K; \tag{2.1}$$

$$\lim_n \int h(f_n - g_n)dx = 0 \qquad \text{for all } h \in C_K. \tag{2.2}$$

PROOF. Let $h \in C_K$. We have

$$\int h(x)(f_n(x) - g_n(x))dx = \int \varphi(t)d(F_n(t) - G_n(t)) \tag{2.3}$$

where $\varphi(t) = \int h(x)e^{itx}dx$. The function $\varphi \in C_0$, since it is the Fourier transform of an L^1 function. Since the total variation of $F_n - G_n$ is bounded by 2 the condition (2.1) implies that it actually holds for all $h \in C_0$. Therefore (2.1) implies (2.2). Now the Fourier transforms of continuous functions with compact support are dense in C_0. Therefore (2.3) shows that (2.2) implies that (2.1) holds for a dense subset of C_0. By obvious approximation argument we conclude that (2.1) holds for all $h \in C_0$.

The following proposition also follows by similar argument.

PROPOSITION 2.2. *Let (F_n) be a sequence of s.p.d.f.s and (f_n) the corresponding sequence of ch.f.s. The following two conditions are equivalent:*

$$\lim_{m,n\to\infty} \int hd(F_n - F_m) = 0 \qquad \text{for all } h \in C_K; \tag{2.4}$$

$$\lim_{m,n\to\infty} \int h(f_n - f_m)dx = 0 \qquad \text{for all } h \in C_K. \tag{2.5}$$

3. Vague convergence of convolutions

The basic reason why vague convergence is less tractable than complete convergence in connection with problems involving convolutions is that $F_n \to F$ and $G_n \to G$ does not imply $F_n * G_n \to F * G$. If $F_n = \delta_n$, $G_n = \delta_{-n}$, then $F_n \to 0$, $G_n \to 0$, but $F_n * G_n \to \delta_0$. To conclude that $F_n \to F$ and $G_n \to G$ implies $F_n * G_n \to F * G$ some supplementary conditions are clearly required. The complete convergence of one of the sequences is enough. In the next proposition we give a more general condition which will be useful.

PROPOSITION 3.1. *If $F_n \to F$ and $G_n \to G$, and for each $a > 0$*

$$\lim_{|\lambda| \to \infty} (G_n(\lambda + a) - G_n(\lambda)) = 0, \qquad \text{uniformly in } n, \tag{3.1}$$

then $F_n * G_n \to F * G$.

Note that $G_n \xrightarrow{c} G$ implies (3.1).

PROOF. Let $B = \{(x, y): a \leqq x \leqq b, c \leqq y \leqq d\}$. If $F(\{a, b\}) = 0$, $G(\{c, d\}) = 0$, then it is easily seen that for a bounded continuous function h on R^2

$$\iint_B h(x, y) dF_n(x) dG_n(y) \to \iint_B h(x, y) dF(x) dG(y). \tag{3.2}$$

Let g be a continuous function with compact support on R^1, then

$$\begin{aligned} \int g(z) dF_n * G_n(z) &= \int \int g(x + y) dF_n(x) dG_n(y) \\ &= \iint_{|x| \leqq \lambda} g(x + y) dF_n(x) dG_n(y) \\ &\quad + \iint_{|x| > \lambda} g(x + y) dF_n(x) dF_n(y). \end{aligned}$$

If the support of g is contained in $[-a, a]$ we can take $\lambda \to \infty$ so that (3.2) applies to the rectangle $|x| \leqq \lambda$, $|y| \leqq \lambda + a$ with $g(x + y) = h(x, y)$. For such λ

$$\iint_{|x| \leqq \lambda} g(x + y) dF_n(x) dG_n(y) \to \iint_{|x| \leqq \lambda} g(x + y) dF(x) dG(y)$$

and taking $|g| \leqq 1$ we also have

$$\int_{|x| > \lambda} \left\{ \int |g(x + y)| dG_n(y) \right\} dF_n(x) \leqq \int_{|x| > \lambda} (G_n(x + a) - G_n(x - a)) dF_n(x).$$

By (3.1) the last expression is uniformly small in n if λ is chosen big. This proves the proposition.

4. Compactness and normalization

For a sequence (F_n) of p.d.f.s the Helly selection principle asserts the existence of a vaguely convergent subsequence. The tightness condition

$$\lim_{\lambda \to \infty} (1 - F_n(\lambda) + F_n(-\lambda)) = 0, \qquad \text{uniformly in } n, \tag{4.1}$$

is equivalent to the assertion that every vaguely convergent subsequence is completely convergent. Frequently one is interested not only in the given sequence (F_n) but in normalized sequences $G_n(x) = F_n(a_n x + b_n)$, $a_n > 0$, b_n real. It is well-known that if $F_n \xrightarrow{c} U$, and U is nondegenerate, then $F_n(a_n x + b_n)$ converges completely to a nondegenerate p.d.f. if and only if $a_n \to a > 0$, $b_n \to b$. (See Feller [2], VIII. 2, lemma 1.) This result is false for vague convergence. We will prove the following proposition which will be needed later.

PROPOSITION 4.1. *If $X_1, X_2, \cdots$ is a sequence of independent, identically distributed, r.v.s and $S_n = \Sigma_{k=1}^n X_k$, and if $\mathscr{L}(S_n/a_n) \to U \neq 0$, U not concentrated in the origin, then $a_n \to \infty$ and $a_{n+1}/a_n \to 1$ as $n \to \infty$.*

PROOF. If $a_n \to \infty$, $X_n/a_n \to 0$ in probability and one obtains at once that $a_{n+1}/a_n \to 1$. We now verify that $a_n \to \infty$. If (1.3) holds for some $\lambda > 0$, then by Proposition 1.1 we have $\mathrm{E}\, X_1 = 0$ and $\mathrm{E}\, X_1^2 < \infty$ so that $n^{-1/2} S_n$ converges completely to $N(0, \sigma^2)$, $\sigma^2 > 0$. Now if (a_n) is bounded along (n_k), then along this subsequence, for $\gamma > 0$

$$\mathrm{P}\{|S_n/a_n| > \gamma\} = \mathrm{P}\{|n^{-1/2} S_n| > \gamma a_n n^{-1/2}\} \to 1$$

since $a_{n_k}/n_k^{1/2} \to 0$, but this contradicts the hypothesis that $\mathscr{L}(S_n/a_n) \to U$. Now assume that (1.3) does not hold for any $\lambda > 0$. Then, if (a_{n_k}) is bounded, for n_k large

$$\mathrm{P}\{|n_k^{-1/2} S_{n_k}| < \lambda\} \geqq \mathrm{P}\{|a_{n_k}^{-1} S_{n_k}| < \lambda\},$$

which again contradicts our hypothesis.

We will also need the following proposition.

PROPOSITION 4.2. *Let (F_n) be a sequence of p.d.f.s, $F_n \xrightarrow{c} F$, F nondegenerate. Suppose $F_n(a_n x + b_n) \to G \neq 0$, $\liminf a_n > 0$, then $\|G\| = 1$, and if G is nondegenerate then $G(x) = F(ax + b)$, $a_n \to a$ and $b_n \to b$.*

PROOF. If $a_n \to \infty$ along n_k, then $F_n(a_n x) \to \delta_0$ along n_k since (F_n) is a tight sequence. Therefore $F_n(a_n x + b_n) \to G \neq 0$ along n_k if and only if b_n/a_n converges to a finite limit along n_k. Thus G is degenerate in this case and $\|G\| = 1$. We may therefore assume $0 < \delta \leqq a_n \leqq A < \infty$. In this case $F_n(a_n x)$ is a tight sequence, hence $F_n(a_n x + b_n) \to G \neq 0$ implies (b_n) must remain bounded. Therefore $F_n(a_n x + b_n)$ is a tight sequence and its vague convergence to G is complete convergence. The rest follows from our remarks before Proposition 4.1.

5. Vague convergence of infinitely divisible laws

We recall that a p.d.f. F with ch.f. φ is infinitely divisible if and only if

$$\log \varphi(u) = iu\alpha + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\, d\Psi(x) \tag{5.1}$$

where α is real and Ψ is a nonnegative multiple of a p.d.f. We shall write

$$\log \varphi = (\alpha, \Psi), \qquad F = [\alpha, \Psi]. \tag{5.2}$$

A sequence $([\alpha_n, \Psi_n])$ of infinitely divisible p.d.f.s converges completely if and only if $\alpha_n \to \alpha$, $\Psi_n \xrightarrow{c} \Psi$, and then the limit is $[\alpha, \Psi]$. Nothing that nice holds for vague convergence. However, we get the following proposition.

PROPOSITION 5.1. *Let $F_n = [\alpha_n, \Psi_n]$, $n \geqq 1$, be a sequence of infinitely divisible p.d.f.s. If there exists an infinitely divisible p.d.f. $F = [\alpha, \Psi]$ such that*

$$\alpha_n \to \alpha, \tag{5.3}$$

$$\Psi_n \to \Psi, \tag{5.4}$$

$$\|\Psi_n\| \to \lambda_0 < \infty, \tag{5.5}$$

and given $a > 0$, as $\lambda \to \infty$ (or $\lambda \to -\infty$)

$$\Psi_n(\lambda + a) - \Psi_n(\lambda) \to 0, \qquad \textit{uniformly in } n, \tag{5.6}$$

then

$$F_n \to e^{-\gamma} F, \tag{5.7}$$

where $\gamma = \lambda_0 - \|\Psi\|$. If (5.6) is not assumed and $F_n \to H$, then $H([a, b]) \geqq e^{-\gamma} F((a, b))$ for each finite interval (a, b).

PROOF. For $\beta > 0$ we define Ψ_n^β by

$$\Psi_n^\beta(A) = \Psi_n(A \cap (\beta, \infty)) \tag{5.8}$$

for a Borel set A. $\Psi_n^{-\beta}$ and $\bar{\Psi}_n^\beta$ are defined similarly by replacing (β, ∞) by $(-\infty, -\beta)$ and $[-\beta, \beta]$, respectively, in (5.8). We then have

$$[\alpha_n, \Psi_n] = [0, \Psi_n^{-\beta}] * [\alpha_n, \bar{\Psi}_n^\beta] * [0, \Psi_n^\beta]. \tag{5.9}$$

If $-\beta$, β are points of continuity of Ψ then clearly $[\alpha_n, \bar{\Psi}_n^\beta] \xrightarrow{c} [\alpha, \bar{\Psi}^\beta]$, where Ψ^β, $\Psi^{-\beta}$ and $\bar{\Psi}^\beta$ are defined analogously to Ψ_n as in (5.8). We can thus find $\beta_n \to \infty$ such that $\pm\beta_n$ are points of continuity of Ψ and

$$[\alpha_n, \bar{\Psi}_n^{\beta_n}] \xrightarrow{c} [\alpha, \Psi]. \tag{5.10}$$

The following lemma will be used to finish the proof.

LEMMA 5.2. *Under the conditions of Proposition* 5.1, *writing* $G_n^\beta = [0, \Psi_n^\beta]$, *and assuming* (5.6) *for* $\lambda \to \infty$, *for* $a > 0$ *we have*

$$\lim_{\beta \to \infty} G_n^\beta(t + a) - G_n^\beta(t) = 0, \qquad \textit{uniformly in } n \textit{ and } t; \tag{5.11}$$

and for $\beta > 0$

$$G_n^\beta(x) = \begin{cases} \exp(-\lambda_n(\beta)), & m_n(\beta) < x < \beta + m_n(\beta), \\ 0, & x < m_n(\beta), \end{cases} \tag{5.12}$$

where

$$\lambda_n(\beta) = \int_{-\infty}^{\infty} \frac{1+x^2}{x^2} d\Psi_n^\beta(x), \qquad m_n(\beta) = \int_{-\infty}^{\infty} \frac{x}{1+x^2} d\Psi_n^\beta(x). \tag{5.13}$$

Analogues of (5.12) *and* (5.13) *hold for* $\beta < 0$ *as well.*

PROOF OF LEMMA. Let

$$dH_n^\beta(x) = \lambda_n(\beta)^{-1}(1 + x^2)x^{-2} d\Psi_n^\beta(x).$$

Then $H_n \equiv H_n^\beta$ is a p.d.f., and

$$G_n^\beta(x - m_n(\beta)) = \exp(-\lambda_n(\beta)) \sum_{k=0}^{\infty} \frac{\lambda_n(\beta)^k}{k!} H_n^{*k}(x). \tag{5.14}$$

(5.12) follows from this immediately. By (5.5) we have

$$\lim_{\beta\to\infty} m_n(\beta)=0, \qquad \text{uniformly in } n. \tag{5.15}$$

For $t>0$, $a>0$, β large (so $m_n(\beta)\leqq 1$)

$$\begin{aligned} G_n^\beta(t+a)-G_n^\beta(t) &\leqq G_n^\beta(t+a+1-m_n(\beta))-G_n^\beta(t-m_n(\beta)) \\ &= e^{-\lambda_n(\beta)}\sum_{k=0}^\infty \frac{\lambda_n(\beta)^k}{k!}[H_n^{*k}(t+a+1)-H_n^{*k}(t)] \\ &\leqq e^{-\lambda_n(\beta)}\sum_{k=0}^\infty \frac{\lambda_n(\beta)^k}{k!}Q_{H_n}(2a+2). \end{aligned}$$

By the definition of H_n^β we have

$$Q_{H_n}(2a+2)\leqq \lambda_n(\beta)^{-1}\frac{1+\beta^2}{\beta^2}\sup_{x>\beta}(\Psi_n(x+a+1)-\Psi_n(x)).$$

Therefore

$$G_n^\beta(t+a)-G_n^\beta(t)\leqq e^{-\lambda_n(\beta)}(e^{\lambda_n(\beta)}-1)\frac{1}{\lambda_n(\beta)}\frac{1+\beta^2}{\beta^2}\sup_{x>\beta}(\Psi_n(x+a+1)-\Psi_n(x)).$$

This together with (5.6) implies (5.11).

To finish the proof of the proposition we will apply the lemma with $\beta>0$ and its analogue with $\beta<0$. We have

$$\Psi_n([\beta_n,\infty))\leqq \lambda_n(\beta_n)\leqq \frac{1+\beta_n^2}{\beta_n^2}\Psi_n([\beta_n,\infty)), \tag{5.16}$$

and a similar inequality holds for $-\beta_n$. We now pick $\beta_n\to\infty$ such that (5.10) holds. By (5.5) and (5.16) every sequence has a further subsequence such that

$$\lambda_n(\beta_n)\to\gamma_1, \qquad \lambda_n(-\beta_n)\to\gamma_2, \tag{5.17}$$

where the limits individually may depend on the subsequence but they satisfy

$$\gamma_1+\gamma_2=\lambda_0-\|\Psi\|. \tag{5.18}$$

It thus follows from (5.12) that every sequence has a further subsequence along which

$$G_n^{\beta_n}\to e^{-\gamma_1}\delta_0, \qquad G_n^{-\beta_n}\to e^{-\gamma_2}\delta_0. \tag{5.19}$$

We now apply (5.11) to use Proposition 3.1 to conclude

$$G_n^{\beta_n}*G_n^{-\beta_n}\to e^{-(\gamma_1+\gamma_2)}\delta_0=e^{-\gamma}\delta_0. \tag{5.20}$$

One more application of Proposition 3.1 shows that every sequence has a further subsequence along which

$$G_n^{\beta_n} * [\alpha_n, \bar{\Psi}_n^{\beta_n}] * G_n^{-\beta_n} \to e^{-\gamma}\delta_0 * [\alpha, \Psi],$$

and this means the whole sequence converges to the desired limit. This proves the first part of the proposition. For the second part, if $a < b$, then, writing $\bar{G}_n^{\beta_n} = [\alpha_n, \bar{\Psi}_n^{\beta_n}]$, we have

$$F_n([a,b]) \geqq \bar{G}_n^{\beta_n}([a+\varepsilon, b-\varepsilon]) G_n^{\beta_n}\left(\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]\right) G_n^{-\beta_n}\left(\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]\right).$$

(5.10) and (5.19) still hold even if (5.6) does not, therefore

$$\liminf_n F_n([a,b]) \geqq F([a+\varepsilon, b-\varepsilon])e^{-\gamma}$$

provided $a+\varepsilon$ and $b-\varepsilon$ are continuity points of F. It follows that

$$H([a,b]) \geqq e^{-\gamma}F((a,b)).$$

The following result is now obvious.

THEOREM 5.3. *Let $F_n = [\alpha_n, \Psi_n]$, $n \geqq 1$, be a sequence of infinitely divisible p.d.f.s. Suppose (α_n) and $(\|\Psi_n\|)$ are bounded sequences and condition* (5.6) *holds. Then (F_n) converges vaguely if and only if $\alpha_n \to \alpha$, $\Psi_n \to \Psi$, $\|\Psi_n\| \to \|\Psi\| + \gamma$, and then $F_n \to e^{-\gamma}[\alpha, \Psi]$.*

REMARK 5.4. The condition (5.6) clearly holds if the measures Ψ_n are all concentrated on a half-line $[a, \infty)$ (or on $(-\infty, a]$).

The class of infinitely divisible p.d.f.s is closed under complete convergence. The closure under vague convergence is not precisely known. The next theorem shows it is extensive.

THEOREM 5.5. *For any p.d.f. H and any $\beta \in [0, e^{-1}]$, βH is the vague limit of a sequence of infinitely divisible p.d.f.s. The set of all such β is a closed interval.*

PROOF. Let $F_n(x) = H(x+n)$, and for $\lambda \geqq 0$

$$\psi_n(u) = \lambda \int_{-\infty}^{\infty} (e^{iux} - 1)dF_n(x).$$

Then

$$\varphi_n(u) = \exp\{\psi_n(u) + iun\}$$

is the ch.f. of a p.d.f. G_n which is given by

$$G_n(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} F_n^{*k}(x-n).$$

Evidently

$$F_n^{*k} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad k = 0, 1, 2, \cdots$$

and

$$G_n \to \lambda e^{-\lambda} H.$$

Since the closure of the range of $\lambda e^{-\lambda}$ is $[0, e^{-1}]$, the first part of the theorem is proved.

REMARK 5.6. (a) Evidently a p.d.f. H will be the vague limit of a sequence of infinitely divisible p.d.f.s if and only if H is infinitely divisible. For any p.d.f. H there exists a compact interval K_H such that βH is in the vague closure of infinitely divisible p.d.f.s if and only if $\beta \in K_H$. Theorem 5.5 says that $[0, e^{-1}] \subset K_H$. Example 5.7 below shows that $\max K_H$ can be as close to 1 as desired.

(b) The proof of Theorem 5.5 shows that for $[\alpha_n, \Psi_n]$ to converge vaguely (to nonzero distributions) the sequence (α_n) need not stay bounded. If one demands that (α_n) stay bounded then the class of vague limits is smaller. In this connection the following example is of interest.

EXAMPLE 5.7. Let $\lambda > 0$ and $\Psi_n = \lambda\delta_{n+1} + \lambda\delta_{-n}$. Then $[0, \Psi_n] \to F$, where F is concentrated on $\{0, 1, 2, \cdots\}$, $F(\{0\}) = e^{-2\lambda}$, $F(\{k\}) = e^{-2\lambda}\lambda^{2k}/(k!)^2$. F is not a constant multiple of an infinitely divisible p.d.f. Also $\|F\|$ can be made as close to 1 as desired by taking λ small.

6. Triangular arrays

Consider the classical setup of triangular arrays: X_{nk}, $1 \leqq k \leqq k_n$, $n = 1, 2, \cdots$, $\mathscr{L}(X_{nk}) = F_{nk}$. The random variables in each row, indexed by n, are assumed independent. The array is called *infinitesimal* if given $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{1 \leqq k \leqq k_n} \mathrm{P}[|X_{nk}| > \varepsilon] = 0. \tag{6.0}$$

Let $S_n = \sum_{k=1}^{k_n} X_{nk}$ denote the n-th row sum. An excellent exposition of the classical results on complete convergence of $\mathscr{L}(S_n)$ is given in Gnedenko and Kolmogorov [3]; our emphasis here is however closer to that found in the discussion by Feller [2]. In the following subsections we will discuss the vague convergence of $\mathscr{L}(S_n)$.

6a. *Centering of triangular arrays.* Let (X_{nk}) be an infinitesimal triangular array. In working with such arrays it is frequently convenient to center them by introducing constants α_{nk} and working with $(X_{nk} - \alpha_{nk})$. The traditional choice for the α_{nk} are the truncated means. In [2] Feller observed that the α_{nk} should be picked to satisfy the relation (6.2) below. It turns out that in studying vague convergence such a choice is very useful. Somewhat surprisingly the traditional choice of truncated means may not fulfill this condition. However, choosing the α_{nk} so that $X_{nk} - \alpha_{nk}$ has zero truncated mean, for some truncation point, is a good choice, as we will show.

For $c > 0$ define

$$\beta_{nk}(c) = \int_{|x| \leqq c} x dF_{nk}(x). \tag{6.1}$$

The array (X_{nk}) is said to be *centered* if for some $c > 0$

$$\zeta_n(c) = \frac{\sum_{k=1}^{k_n} \beta_{nk}(c)^2}{\sum_{k=1}^{k_n} \int_{|x| \leqq c} x^2 dF_{nk}(x)} \to 0 \qquad \text{as } n \to \infty. \tag{6.2}$$

Lemma 6.1 shows that if the array is centered for some c, then it is centered for all larger c, while Lemma 6.3 shows that one can always find constants d_{nk} such that $(X_{nk} - d_{nk})$ is a centered array.

Lemma 6.1. *If for the infinitesimal array* (X_{nk}) *the condition* (6.1) *holds for some* $c > 0$, *then it holds for all* $c' \geqq c$.

Proof. We have

$$\begin{aligned}\beta_{nk}(c')^2 &= \left(\beta_{nk}(c) + \int_{c<|x| \leqq c'} x dF_{nk}\right)^2 \\ &\leqq 2\beta_{nk}(c)^2 + 2\left(\int_{c<|x| \leqq c'} x dF_{nk}\right)^2 \\ &\leqq 2\beta_{nk}(c)^2 + 2\int_{c<|x| \leqq c'} x^2 dF_{nk}\, \mathrm{P}[|X_{nk}| > c].\end{aligned}$$

Therefore

$$\sum_k \beta_{nk}(c')^2 \leqq 2\sum_k \beta_{nk}(c)^2 + 2\theta_n\delta_n$$

where

$$\theta_n = \max_k \mathrm{P}[|X_{nk}| > c], \qquad \delta_n = \sum_k \int_{c<|x|\leqq c'} x^2 dF_{nk}.$$

We thus have

$$\zeta_n(c') \leqq 2\zeta_n(c) + 2\theta_n.$$

Since $\theta_n \to 0$ by infinitesimality, the lemma is proved.

REMARK 6.2. (a) It is not true that if (6.2) holds for some $c > 0$, then it holds for all $c > 0$.

(b) Surprisingly, it is not true that if (X_{nk}) is an infinitesimal array then $(X_{nk} - \beta_{nk})$ is necessarily centered. Consider X_{nk}, $1 \leqq k \leqq n$, $n = 1, 2, \cdots$ with $\mathrm{P}[X_{nk} = n] = n^{-1}$, $\mathrm{P}[X_{nk} = n^{-3}] = 1 - n^{-1}$.

(c) Constants δ_{nk} can be found such that $(X_{nk} - \delta_{nk})$ is centered and $\int_{|x|\leqq c} x dF_{nk}(x + \delta_{nk}) = 0$.

LEMMA 6.3. *If (X_{nk}) is an infinitesimal triangular array and constants d_{nk} are defined by*

$$\int_{|x|\leqq c} (x - d_{nk}) dF_{nk}(x) = 0 \tag{6.3}$$

for some $c > 0$, then the array $(X_{nk} - d_{nk})$ is a centered infinitesimal array for which the analogue of (6.2) *holds for all $c' > c$.*

PROOF. Let $X'_{nk} = X_{nk} - d_{nk}$ and $F'_{nk}(x) = F_{nk}(x + d_{nk})$. Let ζ'_n be defined by (6.2) in terms of F'_{nk}. We need to show that $\zeta'_n(c') \to 0$ as $n \to \infty$ for all $c' > c$. Since (X_{nk}) is infinitesimal, $d_{nk} \to 0$ uniformly in k as $n \to \infty$, therefore (X'_{nk}) is also infinitesimal. We have

$$|\beta'_{nk}(c)| = \left| \int_{|x - d_{nk}|\leqq c} (x - d_{nk}) dF_{nk}(x) \right|$$

and using (6.3) we get

$$|\beta'_{nk}(c)| \leqq \int_{[-c, -c + d_{nk}] \cup [c, c + d_{nk}]} |x - d_{nk}| dF_{nk}(x)$$

where $[a, b]$ is to be interpreted as $[b, a]$ if $b < a$. Hence for n sufficiently large

$$\beta'_{nk}(c)^2 \leqq 16c^2\gamma_{nk}^2$$

where

$$\gamma_{nk} = F_{nk}([-c, -c + d_{nk}] \cup [c, c + d_{nk}]).$$

If $c' > c$, then for n sufficiently large, we also get

$$\int_{|x| \leqq c'} x^2 dF'_{nk}(x) \geqq \frac{c^2}{4} \gamma_{nk}.$$

Since $\gamma_{nk} \to 0$ uniformly in k it follows that

$$\frac{\sum_k (\beta'_{nk}(c))^2}{\sum_k \int_{|x| \leqq c'} x^2 dF'_{nk}(x)} \leqq 64 \frac{\sum_k \gamma_{nk}^2}{\sum_k \gamma_{nk}} \to 0$$

as $n \to \infty$. To finish the proof observe that

$$\frac{\sum_k \left(\int_{c<|x| \leqq c'} x dF'_{nk}\right)^2}{\sum_k \int_{|x| \leqq c'} x^2 dF'_{nk}} \leqq \frac{c^2 \sum_k (\mathrm{P}[c < |X'_{nk}| \leqq c'])^2}{c^2 \sum_k \mathrm{P}[c < |X'_{nk}| \leqq c']} \to 0$$

as $n \to \infty$. Therefore $\zeta'_n(c') \to 0$ for $c' > c$.

The next proposition gives a tightness criterion for centered arrays that will be needed.

PROPOSITION 6.3a. *Let (X_{nk}) be a centered infinitesimal array and S_n its n-th row sum. Then $(\mathscr{L}(S_n))$ is a tight sequence if and only if we have*

$$(\alpha) \qquad \sup_n \sum_k \int \frac{x^2}{1+x^2} dF_{nk}(x) < \infty$$

and

$$(\beta) \qquad \lim_{\lambda \to \infty} \sum_k \mathrm{P}[|X_{nk}| > \lambda] = 0 \qquad \textit{uniformly in } n.$$

PROOF. The proof is essentially the same as the proof of lemma 1 [2], p. 299. The condition (7.3) in [2], p. 229, becomes the condition that for some $c > 0$

$$\sup_n \left[\sum_k \int_{|x| \leqq c} x^2 dF_{nk}(x) - \sum_k \left(\int_{|x| \leqq c} x dF_{nk}\right)^2\right] < \infty.$$

Since the array is centered, we can find $c > 0$ so that this condition is equivalent to the condition

$$\sup_n \sum_k \int_{|x| \leq c} x^2 dF_{nk}(x) < \infty.$$

This condition and (β) are clearly equivalent to (α) and (β). But (β) corresponds to the condition (7.4) [2], p. 299. This completes the proof.

6b. *The dissipative property.* Let (F_n) be a sequence of p.d.f.s. The sequence (F_n) is called *dissipative* if

$$F_n(\cdot + b_n) \to 0, \qquad \text{for every real sequence } (b_n). \tag{6.4}$$

A sequence of random variables (X_n) will be called dissipative if the sequence $(\mathscr{L}(X_n))$ is dissipative. Note that (6.4) is equivalent to

$$\lim_n Q_{F_n}(y) = 0, \qquad \text{for every real } y. \tag{6.5}$$

We now give a necessary and sufficient condition for a triangular array to be dissipative. The result will be used later.

Proposition 6.4. *Let (X_{nk}) be a centered infinitesimal array and let S_n be the n-th row sum. Then $(\mathscr{L}(S_n))$ is dissipative if and only if*

$$\lim_n \sum_{k=1}^{k_n} \int \frac{x^2}{1+x^2} dF_{nk}(x) = \infty. \tag{6.6}$$

This proposition will also be used in the form of the following corollary which is an immediate consequence.

Corollary 6.5. *Let (X_{nk}) and S_n be as in the above proposition. Then $(\mathscr{L}(S_n))$ does not possess a dissipative subsequence if and only if*

$$\sup_n \sum_{k=1}^{k_n} \int \frac{x^2}{1+x^2} dF_{nk}(x) < \infty. \tag{6.7}$$

Proof of Proposition 6.4. Assume (6.6). Let ${}^\circ F_{nk}$ be the symmetrization of F_{nk}. Then for $c > 0$

$$\int_{|x|\leqq c} x^2 d^\circ F_{nk}(x) \geqq \int_{\substack{|x|\leqq c/2 \\ |y|\leqq c/2}} |x-y|^2 dF_{nk}(x) dF_{nk}(y)$$

$$= 2\int_{\substack{|x|\leqq c/2 \\ |y|\leqq c/2}} x^2 dF_{nk}(x) dF_{nk}(y) - 2\left(\int_{|x|\leqq c/2} x dF_{nk}(x)\right)^2.$$

Since the array is centered, for some $c > 0$ we have

$$\sum_k \left(\int_{|x|\leqq c/2} x dF_{nk}(x)\right)^2 = o\left(\sum_k \int_{|x|\leqq c} x^2 dF_{nk}(x)\right).$$

Therefore, using infinitesimality, we have for some $c > 0$ and all n large

$$\sum_k \int_{|x|\leqq c/2} x^2 dF_{nk}(x) \leqq \sum_k \int_{|x|\leqq c} x^2 d^\circ F_{nk}(x).$$

It follows that (6.6) implies

$$\lim_n \sum_k \int \frac{x^2}{1+x^2} d^\circ F_{nk}(x) = \infty$$

which is equivalent to

$$\lim_n \left(\sum_k \left[\int_{|x|\leqq c} x^2 d^\circ F_{nk}(x) + c^2 \int_{|x|>c} d^\circ F_{nk}(x) \right]\right) = \infty \tag{6.8}$$

for all $c > 0$. Therefore the concentration function inequality (1.2) implies that $\mathscr{L}(S_n)$ is dissipative.

Now assume that (6.6) does not hold. Dropping to a subsequence, if necessary, we assume that

$$\sup_n \sum_k \int \frac{x^2}{1+x^2} dF_{nk}(x) < \infty. \tag{6.9}$$

We will show that this implies $\mathscr{L}(S_n)$ is not dissipative. Let $c > 0$ such that (6.2) holds and let

$$X'_{nk} = \begin{cases} X_{nk} & \text{if } |X_{nk}| \leqq c, \\ 0 & \text{if } |X_{nk}| > c, \end{cases}$$

$$X''_{nk} = X_{nk} - X'_{nk}.$$

$S'_n = \Sigma_k X'_{nk}$ and $S''_n = \Sigma_k X''_{nk}$. For $\lambda > 0$ we have

$$\begin{aligned} \mathrm{P}[|S_n| \leqq \lambda,\ S''_n = 0] &= \mathrm{P}[|S'_n| \leqq \lambda,\ S''_n = 0] \\ &\geqq \mathrm{P}[S''_n = 0] - \mathrm{P}[|S'_n| > \lambda] \\ &\geqq \mathrm{P}[S''_n = 0] - \frac{\mathrm{E}(S'^2_n)}{\lambda^2} \end{aligned}$$

by Chebychev's inequality. Now $\mathrm{E}(S'^2_n) = \mathrm{Var}(S'_n) + (\mathrm{E}\, S'_n)^2 \leqq 2\Sigma_k\, \mathrm{E}\, X'^2_{nk}$ for all large n since the array is centered. Therefore by (6.9) we have $\sup_n \mathrm{E}(S'^2_n) < \infty$, and for λ large

$$\mathrm{P}[|S_n| \leqq \lambda] \geqq \frac{1}{2}\mathrm{P}[S''_n = 0] = \frac{1}{2}\prod_k \mathrm{P}[|X_{nk}| > c] \tag{6.10}$$

and the last quantity in (6.10) is bounded away from zero as $n \to \infty$ because $\sup_n \Sigma_k\, \mathrm{P}[|X_{nk}| > c] < \infty$ by (6.9). Thus $\mathcal{L}(S_n)$ cannot be dissipative.

We will need the following lemma. The condition (6.12) which we find below occurs frequently in classical work on the central limit problem; Theorem 6.7 indicates why this should be so.

LEMMA 6.6. *Let (F_{nk}) be an infinitesimal triangular array. Let (β_{nk}) be constants satisfying (6.1) and let (d_{nk}) be defined by (6.3); both sequences are defined for the same $c > 0$. Then the following two conditions are equivalent:*

$$\sup_n \sum_k \int \frac{x^2}{1+x^2}\, dF_{nk}(x + d_{nk}) < \infty; \tag{6.11}$$

$$\sup_n \sum_k \int \frac{x^2}{1+x^2}\, dF_{nk}(x + \beta_{nk}) < \infty. \tag{6.12}$$

PROOF.

$$\begin{aligned} \int \frac{x^2}{1+x^2}\, dF_{nk}(x + d_{nk}) &= \int \frac{(x + \beta_{nk} - d_{nk})^2}{1 + (x + \beta_{nk} - d_{nk})^2}\, dF_{nk}(x + \beta_{nk}) \\ &\leqq 2\int \frac{x^2}{1 + (x + \beta_{nk} - d_{nk})^2}\, dF_{nk}(x + \beta_{nk}) + 2(\beta_{nk} - d_{nk})^2. \end{aligned}$$

Since $\beta_{nk} \to 0$, $d_{nk} \to 0$ uniformly in k as $n \to \infty$, the first term to the right of the inequality can be dominated by

$$4\int_{|x|>\varepsilon} \frac{x^2}{1+x^2}\, dF_{nk}(x + \beta_{nk}) + 2\int_{|x|\leqq\varepsilon} x^2 dF_{nk}(x + \beta_{nk})$$

for $\varepsilon > 0$ and n sufficiently large. By (6.12) this last estimate summed on k is bounded in n. We now look at $(\beta_{nk} - d_{nk})^2$:

$$(\beta_{nk} - d_{nk})^2 = d_{nk}^2 \theta_{nk}^2,$$

where

$$\begin{aligned} \theta_{nk} &= \int_{|x|>c} dF_{nk}(x) \\ &= \int_{|x+\beta_{nk}|>c} dF_{nk}(x+\beta_{nk}) \\ &\leqq \int_{|x|>c/2} dF_{nk}(x+\beta_{nk}) \end{aligned}$$

for n sufficiently large, uniformly in k. Therefore again by (6.12) we have

$$\sup_n \sum_k d_{nk}^2 \theta_{nk}^2 < \infty.$$

Hence (6.12) implies (6.11). The same argument with the roles of β_{nk} and d_{nk} switched shows that (6.11) implies (6.12) and the lemma is proved.

THEOREM 6.7. *Let (X_{nk}) be an infinitesimal triangular array and let S_n be the n-th row sum. Then $\mathscr{L}(S_n)$ possesses no dissipative subsequence if and only if* (6.12) *holds.*

PROOF. Let β_{nk} and d_{nk} be defined as in Lemma 6.6. By Lemma 6.3 the array $(X_{nk} - d_{nk})$ is centered. $\mathscr{L}(S_n)$ does not possess a dissipative subsequence if and only if $\mathscr{L}(S_n - \Sigma_k d_{nk})$ does not. By Corollary 6.5 this is equivalent to (6.11), which in turn is equivalent to (6.12) by Lemma 6.6. The proof is complete.

The following theorem will now be obtained as a corollary.

THEOREM 6.8. *Let $F_n = [0, \Psi_n]$, $n \geqq 1$, be a sequence of infinitely divisible laws. Then (F_n) is dissipative if and only if $\|\Psi_n\| \to \infty$.*

To avoid centering difficulties we establish the following lemma.

LEMMA 6.9. (a) *(F_n) is dissipative if and only if $({}^\circ F_n)$ is dissipative.*

(b) *If (Ψ_n) is a sequence of measures on the real line and*

$$^*\Psi_n(A) = \frac{1}{2}(\Psi_n(A) + \Psi_n(-A))$$

where $-A = \{x;\ -x \in A\}$, *then*

$$\|\Psi_n\| \to \infty \textit{ if and only if } \|^*\Psi_n\| \to \infty.$$

PROOF. (b) is obvious. Also

$$Q_{F_1 * F_2} \leqq Q_{F_i}, \qquad i = 1, 2$$

where F_1 and F_2 are any p.d.f.s. Therefore if (F_n) is dissipative so is the sequence $({}^\circ F_n)$. If (F_n) is not dissipative then along a subsequence we have

$$\int_{|x-b_n|\leqq a} dF_n(x) \geqq \alpha > 0$$

for a suitable choice of b_n and a. But this implies (along the same subsequence)

$$\int_{|x|\leqq 2a} d^\circ F_n(x) \geqq \alpha > 0$$

hence $({}^\circ F_n)$ is not dissipative.

PROOF OF THEOREM 6.8. Note that ${}^\circ F_n = [0, {}^*\Psi_n]$, where ${}^*\Psi_n$ is defined as in Lemma 6.9. It is clearly enough to prove the theorem for $({}^\circ F_n)$. Let

$${}^\circ F_{nk} = \left[0, \frac{1}{k}{}^*\Psi_n\right].$$

Let

$$\bar{\Psi}_{n,k}(dx) = k\frac{x^2}{Hx^2} d^\circ F_{nk}(x).$$

Then we know (see, e.g. [3], pp. 76–78) that

$$\bar{\Psi}_{n,k}(dx) \xrightarrow{c} {}^*\Psi_n(dx) \qquad \text{as } k \to \infty \tag{6.13}$$

and

$${}^\circ F_{nk}^{*k} = {}^\circ F_n. \tag{6.14}$$

If $\|^*\Psi_n\| \to \infty$, we can find $k_n \to \infty$ such that

$$\|\bar{\Psi}_{n,k_n}\| = k_n \int \frac{x^2}{1+x^2} d^\circ F_{nk_n} \to \infty$$

and the triangular array $(X_{nk}, 1 \leqq k \leqq k_n)$, $\mathscr{L}(X_{nk}) = {}^\circ F_{n,k_n}$, $1 \leqq k \leqq k_n$, $n \geqq 1$, is infinitesimal. It follows from Theorem 6.7 that $({}^\circ F_n)$ is dissipative. Conversely if $({}^\circ F_n)$ is dissipative, we define the infinitesimal triangular array as above. Then by Theorem 6.7 we must have $\|\bar{\Psi}_{n,k_n}\| \to \infty$, but the k_n can be picked so by (6.13) we also have $\|{}^*\Psi_n\| \to \infty$. This finishes the proof.

6c. *Accompanying infinitely divisible laws.* In the theory of complete convergence for triangular arrays the so-called accompanying infinitely divisible laws have been effectively used. Specifically, let (X_{nk}) be an infinitesimal triangular array and let S_n be the n-th row sum. Let f_n be the ch.f. of S_n. It was shown by Gnedenko ([3], p. 112), under assumption (6.12), that there exists a sequence (g_n) of ch.f.s of infinitely divisible laws such that $f_n - g_n \to 0$ as $n \to \infty$, so Proposition 2.1 is applicable. The g_n can be explicitly written down, and Gnedenko's result is a key theorem in the central limit problem. Of course the g_n are not uniquely determined by the requirement $f_n - g_n \to 0$. Indeed, the great convenience of centering is the possibility of choosing g_n in a simpler, more convenient form.

THEOREM 6.10. *Let (X_{nk}) be an infinitesimal triangular array, and (b_n) a sequence of real numbers. Let S_n denote the n-th row sum. If $\mathscr{L}(S_n)$ does not possess a dissipative subsequence, then*

$$\lim_n |\mathrm{E}\, e^{iuS_n} - e^{\Psi_n(u)}| = 0, \quad u \text{ real}, \tag{6.15}$$

where

$$\Psi_n(u) = \sum_k iu\beta_{nk} + \sum_k \int (e^{iux} - 1)dF_{nk}(x + \beta_{nk}) \tag{6.16}$$

and

$$\beta_{nk} = \int_{|x| \leq c} x dF_{nk}(x) \quad \textit{for some } c > 0.$$

In any case, $\mathscr{L}(S_n - b_n) \to F$ if and only if $F_n \to F$, where F_n is the accompanying infinitely divisible law whose ch.f. is given by $\exp(\Psi_n(u) - ib_n u)$.

PROOF. By Theorem 6.7, if $(\mathscr{L}(S_n))$ does not possess a dissipative subsequence then

(6.17) $$\sup_n \sum_k \int \frac{x^2}{1+x^2} dF_{nk}(x+\beta_{nk}) < \infty.$$

If (6.17) holds, then as shown in [3], §24, we get (6.15).

For the second part of the theorem, assume (6.17) first. Then by Proposition 2.1, using (6.15), we conclude that $\mathscr{L}(S_n - b_n) \to F$ if and only if $F_n \to F$. Now assume that (6.17) does not hold. Then by Theorem 6.7 a subsequence $\mathscr{L}(S_{n_j})$ is dissipative, hence $\mathscr{L}(S_{n_j} - b_{n_j})$ is dissipative, and if $\mathscr{L}(S_n - b_n) \to F$ then $F = 0$. We need to show that $F_n \to 0$ in this case. If not, then along a subsequence (m_j) the sequence (F_n) converges to a nonzero limit, in particular (F_{m_j}) is not dissipative. Since $F_n = [\alpha_n, \Psi_n]$ with

$$\|\Psi_n\| = \sum_k \int \frac{x^2}{1+x^2} dF_{nk}(x+\beta_{nk})$$

and α_n suitable constants, it follows from Theorem 6.8 that (6.17) holds along (m_j). But this implies (6.15) along (m_j) by the first part, hence $F_{m_j} \to 0$, a contradiction. Also, if (6.17) does not hold and $F_n \to F$, then by Theorem 6.8 a subsequence (F_{n_j}) is dissipative, therefore $F = 0$. That $\mathscr{L}(S_n - b_n) \to 0$ in this case is proved by contraposition as above except that Theorem 6.7 is applied in place of Theorem 6.8. This finishes the proof.

When (X_{nk}) is centered and infinitesimal, the accompanying laws can be given a simpler form.

THEOREM 6.11. *Let (X_{nk}) be a centered infinitesimal array. If $\mathscr{L}(S_n)$ does not possess a dissipative subsequence then* (6.15) *holds with ψ_n defined by*

(6.18) $$\psi_n(u) = \sum_k \int (e^{iux} - 1) dF_{nk}(x).$$

In any case, $\mathscr{L}(S_n - b_n) \to F$ if and only if $F_n \to F$, where F_n has ch.f. $\exp(\psi_n(u) - ib_n u)$.

PROOF. If $\mathscr{L}(S_n)$ possesses no dissipative subsequence, by Corollary 6.5

(6.19) $$\sup_n \sum_k \int \frac{x^2}{1+x^2} dF_{nk}(x) < \infty.$$

We will show that (6.19) implies

(6.20) $$\lim_n |\mathrm{E}\, e^{iS_n u} - \exp(\psi_n(u))| = 0.$$

The rest of the assertion of the theorem follows by arguments similar to those given in the proof of Theorem 6.10. We will therefore show only that (6.19) implies (6.20). Let φ_{nk} be the ch.f. of F_{nk}. Then it suffices to check that (6.19) implies

$$\lim_n \left| \sum_k \log \varphi_{nk}(u) - \sum_k (\varphi_{nk}(u) - 1) \right| = 0, \quad u \text{ real.} \tag{6.21}$$

We, of course, will need the fact that the array is centered. For convenience we write

$$\alpha_{nk}(u) = \varphi_{nk}(u) - 1.$$

Note that by infinitesimality $\max_k \alpha_{nk} \to 0$ uniformly on bounded intervals, therefore $\log \varphi_{nk}(u)$ is well-defined for n sufficiently large. For u fixed and n sufficiently large

$$\begin{aligned} \left| \sum_k \log \varphi_{nk}(u) - \sum_k \alpha_{nk}(u) \right| &\leqq \sum_{k=1}^{k_n} \sum_{r=2}^{\infty} \frac{|\alpha_{nk}(u)|^r}{r} \\ &\leqq \frac{1}{2} \sum_k |\alpha_{nk}(u)|^2 (1 - |\alpha_{nk}|)^{-1} \\ &\leqq \sum_k |\alpha_{nk}(u)|^2. \end{aligned} \tag{6.22}$$

Now

$$\begin{aligned} |\alpha_{nk}(u)| &= \left| \int (e^{iux} - 1) dF_{nk}(x) \right| \\ &\leqq \left| \int_{|x| \leqq c} (e^{iux} - 1 - iux) dF_{nk}(x) \right| + \left| \int_{|x|>c} (e^{iux} - 1) dF_{nk}(x) \right| \\ &\quad + |u| \left| \int_{|x| \leqq c} x dF_{nk}(x) \right| \\ &\leqq \frac{u^2}{2} \int_{|x| \leqq c} x^2 dF_{nk}(x) + 2 \int_{|x|>c} dF_{nk}(x) + |u| \left| \int_{|x| \leqq c} x dF_{nk}(x) \right|. \end{aligned}$$

For u and c fixed, we denote the sum of the first two terms on the right-side of the last inequality by γ_{nk} and denote the third term by δ_{nk}. Thus

$$|\alpha_{nk}(u)| \leqq \gamma_{nk} + \delta_{nk}.$$

We write

$$\sum_k |\alpha_{nk}(u)|^2 = \sum{}' |\alpha_{nk}|^2 + \sum{}'' |\alpha_{nk}|^2$$

where Σ' is the sum on $[k : \gamma_{nk} \geqq \delta_{nk}]$, and Σ'' is the sum of the rest of the terms.

Therefore

$$\sum_k |\alpha_{nk}(u)|^2 \leqq \max_k |\alpha_{nk}| \sum' |\alpha_{nk}| + 4\sum_k \delta_{nk}^2$$

$$\leqq 2\max_k |\alpha_{nk}| \sum_k \gamma_{nk} + 4\sum_k \delta_{nk}^2.$$

Now $\Sigma_k \gamma_{nk}$ is bounded by (6.19), and $\Sigma_k \delta_{nk}^2 = o(\Sigma_k \gamma_{nk})$ since the array is centered. It follows that $\Sigma_k |\alpha_{nk}(u)|^2 \to 0$ as $n \to \infty$ and (6.21) results from (6.22).

The following corollary is obvious.

COROLLARY 6.12. *The class of s.p.d.f.s. obtained as vague limits of infinitesimal triangular arrays coincides with the class of s.p.d.f.s obtained as vague limits of infinitely divisible p.d.f.s.*

REMARK 6.13. Although our concern is with vague convergence here, it should be noticed that centering brings great convenience in dealing with complete convergence as well. If ψ_n is given by (6.18) then $\exp(\psi_n)$ is the ch.f. of $[\alpha_n, \Psi_n]$ where

$$\alpha_n = \sum_k \int \frac{x}{1+x^2} dF_{nk}(x)$$

and

$$\Psi_n(dx) = \sum_k \frac{x^2}{1+x^2} dF_{nk}(x).$$

By Theorem 6.11 and Proposition 2.1 it follows that $\mathscr{L}(S_n - b_n) \xrightarrow{c} [\alpha, \Psi]$ if and only if $\alpha_n - b_n \to \alpha$ and $\Psi_n \xrightarrow{c} \Psi$. In other words, it eliminates the usual bother at the origin, c.f. condition 2), theorem 1, §25 [3].

7. Independent identically distributed summands under vague convergence

In this section $X_1, X_2, \cdots$ will be independent random variables with a common d.f. F. (a_n) and (b_n) will denote sequences of real numbers with $a_n > 0$, $a_n \to \infty$. We then have an infinitesimal triangular array defined by

$$X_{nk} = \frac{X_k}{a_n}, \qquad 1 \leqq k \leqq n, \quad n \geqq 1. \tag{7.1}$$

As shown in Feller [2], the array is centered if $\mathrm{E}\,X_1$ exists and equals 0 or $\mathrm{E}\,X_1^2 = \infty$. As before we will write

$$S_n = \sum_{k=1}^{n} X_{nk}. \tag{7.2}$$

Necessary and sufficient conditions on F for the existence of sequences (a_n) and (b_n) so that $\mathscr{L}(S_n - b_n)$ converges completely are well known; when such normalizing sequences exist, the limit laws are stable. Under vague convergence (when positive mass is allowed to escape) the situation essentially is that $L(x)$ given by

$$L(x) = 1 - F(x) + F(-x-) \tag{7.3}$$

is slowly varying near infinity as we now proceed to show.

REMARK 7.1. By Theorem 6.11 if the array is centered we may take $[\alpha_n, \Psi_n]$ as the accompanying infinitely divisible laws of S_n, where

$$\alpha_n = n\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dF(a_n x) \tag{7.4}$$

and

$$d\Psi_n(x) = n\frac{x^2}{1+x^2}\,dF(a_n x). \tag{7.5}$$

PROPOSITION 7.2. *Suppose L is slowly varying and along $n_j \nearrow \infty$ we have $n_j L(a_{n_j}) \to \lambda > 0$. Then $\alpha_{n_j} \to 0$ and $\mathscr{L}(S_{n_j}) \to e^{-\lambda}\delta_0$, where α_n is given by* (7.4).

PROOF. For convenience of writing we will prove the proposition for (n) in place of (n_j). The proof is valid along any $n_j \nearrow \infty$. For $c > 1$

$$\begin{aligned}
|\alpha_n| &\leqq n\left(\int_{|x|\leqq c} |x|\,dF(a_n x) + \int_{|x|>c} \frac{|x|}{1+x^2}\,dF(a_n x)\right)\\
&\leqq \frac{n}{a_n}\int_{|x|\leqq ca_n} |x|\,dF(x) + \frac{n}{c}L(ca_n)\\
&= \frac{n}{a_n}\left(-ca_n L(ca_n) + \int_0^{ca_n} L(x)\,dx\right) + \frac{n}{c}L(ca_n)\\
&\to \frac{2\lambda}{c}
\end{aligned}$$

by using the slow variation of L and the fact that $nL(a_n) \to \lambda$. This shows that $\alpha_n \to 0$.

Since L is slowly varying, $\mathrm{E}\,X_1^2=\infty$ and (X_{nk}) is a centered array. We have $\alpha_n \to 0$. Therefore by Remark 7.1 it suffices to show that $[0, \Psi_n] \to e^{-\lambda}\delta_0$, where Ψ_n is given by (7.5). For $0<y<z$

$$\Psi_n((y,z]) + \Psi_n([-z,-y)) \leqq n\int_{(y,z]} dL(a_n x)$$
$$= n[L(a_n y) - L(a_n z)] \to 0,$$

and also

$$\Psi_n([-y,y]) \leqq n\int_{|x|\leqq y} x^2 dF(a_n x)$$
$$= \frac{n}{a_n^2}\int_{|x|\leqq y a_n} x^2 dF(x)$$
$$= \frac{n}{a_n^2}\left[-y^2 a_n^2 L(ya_n) + 2\int_0^{ya_n} yL(y)dy\right]$$
$$\to 0.$$

Therefore $\Psi_n \to 0$. Furthermore for $c>0$

$$\|\Psi_n\| = n\int_{(-\infty,-c)} \frac{x^2}{1+x^2} dF(a_n x) + n\int_{[-c,c]} \frac{x^2}{1+x^2} dF(a_n x)$$
$$+ n\int_{(-c,\infty)} \frac{x^2}{1+x^2} dF(a_n x)$$

and the middle term on the right goes to 0 as $n\to\infty$ by the above argument; the sum of the remaining two terms is near $nL(ca_n)$ uniformly in n. Since $nL(ca_n)\to\lambda$, it follows that $\|\Psi_n\|\to\lambda$. To apply Proposition 5.1 it remains to verify that $\Psi_n(t+l)-\Psi_n(t)\to 0$ uniformly in n as $t\to\infty$ for $l>0$. For $t>0$, $l>0$

$$\Psi_n(t+l) - \Psi_n(t) \leqq \Psi_n((t,t+l]) + \Psi_n([-t-l,-t))$$
$$\leqq n(L(a_n t) - L(a_n(t+l))).$$

For a slowly varying function L the ratio $L(xt)/L(x)\to 1$ uniformly in t on compact intervals as $x\to\infty$; using this fact the last term is seen to tend to 0 uniformly in n as $t\to\infty$. This finishes the proof.

PROPOSITION 7.3. *Suppose* $0<\alpha_1 \leqq a_{n+1}/a_n \leqq \alpha_2 < \infty$ *for all* n, *and that for some* $\lambda>0$ *and real* b_n

$$\mathscr{L}(S_n - b_n) \to e^{-\lambda}\delta_0, \tag{7.6}$$

then L is slowly varying near infinity.

PROOF. The hypotheses imply that $\mathrm{E}X_1^2 = \infty$, hence the array (X_{nk}) is centered. By (7.6) $\mathscr{L}(S_n)$ is not dissipative, therefore by Theorem 6.11 and Remark 7.1 we have

$$[\alpha_n + b_n, \Psi_n] \to e^{-\lambda}\delta_0 \tag{7.7}$$

where α_n and Ψ_n are given by (7.4) and (7.5), respectively. For $\beta > 0$, in the notation of Proposition 5.1, we have

$$[\alpha_n + b_n, \Psi_n^{-\beta}] * [0, \Psi_n^{\beta}] * [0, \bar{\Psi}_n^{\beta}] \to e^{-\lambda}\delta_0. \tag{7.8}$$

We claim that $\|\bar{\Psi}_n^{\beta}\| \to 0$. If not, then along a subsequence $\bar{\Psi}_n^{\beta} \xrightarrow{c} \Psi$ with $\|\Psi\| > 0$ because these measures are concentrated on $[-\beta, \beta]$. Then along a *further* subsequence we have

$$[0, \bar{\Psi}_n^{\beta}] \xrightarrow{c} F$$

and

$$[\alpha + b_n, \Psi_n^{-\beta}] * [0, \Psi_n^{\beta}] \to G.$$

By Proposition 3.1 the convolutions in (7.8) converge to $F * G$ vaguely, where F is a nondegenerate infinitely divisible law. This contradicts the fact that the limit in (7.8) is $e^{-\lambda}\delta_0$. It follows that $\Psi_n \to 0$. In particluar, for $y > 0$

$$n\int_{|x|\leqq y} x^2 dF(a_n x) \to 0. \tag{7.9}$$

Now for $0 \leqq z \leqq y$

$$\begin{aligned} n\int_{|x|\leqq y} x^2 dF(a_n x) &= \frac{n}{a_n^2}\int_{|x|\leqq y a_n} x^2 dF(x) \\ &\geqq \frac{n}{a_n^2}\int_{a_n z < |x| \leqq a_n y} x^2 dF(x) \\ &\geqq \frac{nz^2}{2}(L(a_n z) - L(a_n y)), \end{aligned}$$

hence the last expression tends to zero. Also

$$\|\Psi_n\| = n\int_{|x|\leqq y} \frac{x^2}{1+x^2} dF(a_n x) + n\int_{|x|>y} \frac{x^2}{1+x^2} dF(a_n x). \tag{7.10}$$

If $\|\Psi_n\| \to 0$, then $\Psi_n \xrightarrow{c} 0$ and (7.7) can then hold only if $\alpha_n + b_n \to 0$, $\lambda = 0$, a contradiction. For the same reason $\|\Psi_n\|$ cannot tend to zero along a subsequence. Therefore $\|\Psi_n\| \geqq \alpha > 0$ for all n. By (7.9) and (7.10) we then conclude that since

$$n \int_{|x|>y} \frac{x^2}{1+x^2} dF(a_n x) \leqq nL(a_n y)$$

the quantity $nL(a_n y)$ is bounded away from 0. Therefore, if $0 \leqq z < y$

$$\frac{n(L(a_n z) - L(a_n y))}{nL(a_n y)} \to 0$$

which together with the boundedness condition on a_{n+1}/a_n implies that L is slowly varying near ∞.

PROPOSITION 7.4. *If h is a right-continuous, decreasing function on $[0, \infty)$ which is slowly varying near ∞, and $h(x) \to 0$ as $x \to \infty$, then given $\lambda > 0$ there exist $0 < a_n \nearrow \infty$ such that $nh(a_n) \to \lambda$ as $n \to \infty$.*

PROOF. Let

$$a_n = \inf\{t : h(t) < \lambda / n\}.$$

Then $h(a_n) \leqq \lambda/n$ by right continuity, and $h(a_n -) \geqq \lambda / n$. Since $h(a_n/2)/h(a_n) \to 1$, we have $nh(a_n) \to \lambda$.

PROPOSITION 7.5. *Suppose $\mathcal{L}(S_n - b_n) \to G$ with $0 < \|G\| < 1$. Then there exist constants $c_n > 0$ and b'_n such that $c_n \to \infty$, $c_{n+1}/c_n \to 1$, and $\mathcal{L}(S'_n - b'_n) \to \|G\| \delta_0$, where $S'_n = S_n / c_n$.*

PROOF. Let $\theta = \|G\|$. There exist $n_j \nearrow \infty$, $0 < \varepsilon_j \to 0$, such that for $n \geqq n_j$

$$|(1-\theta) - \mathrm{P}[|S_n - b_n| > j]| \leqq \varepsilon_j \tag{7.11}$$

and

$$|(1-\theta) - \mathrm{P}[|S_n - b_n| > j+1]| \leqq \varepsilon_j. \tag{7.12}$$

Define

$$c'_n = j + \frac{n - n_j}{n_{j+1} - n_j}, \qquad n_j \leqq n \leqq n_{j+1}.$$

Then $c'_n \nearrow \infty$, and $c'_{n+1}/c'_n \to 1$. For $n_j \leqq n \leqq n_{j+1}$

(7.13) $\mathrm{P}[|S_n - b_n| > j+1] \leqq \mathrm{P}[|S_n - b_n| > c'_n] \leqq \mathrm{P}[|S_n - b_n| > j]$.

Let $c_n = c'^{1/2}_n$, $b'_n = b_n/c_n$, then for $\varepsilon > 0$

$$\mathrm{P}[|S'_n - b'_n| \leqq \varepsilon] = \mathrm{P}[|S_n - b_n| \leqq \varepsilon c_n],$$

hence

$$\liminf_n \mathrm{P}[|S'_n - b'_n| \leqq \varepsilon] \geqq \theta.$$

Also, by (7.11)–(7.13) we have for any $a > 0$

$$\liminf_n \mathrm{P}[|S'_n - b'_n| \geqq a] \geqq 1 - \theta.$$

It follows that

$$\mathscr{L}(S'_n - b'_n) \to \theta\delta_0.$$

THEOREM 7.6. (i) *Suppose* $0 < \alpha_1 \leqq a_{n+1}/a_n \leqq \alpha_2 < \infty$, *and for some* (b_n)

(7.14) $$\mathscr{L}(S_n - b_n) \to G, \qquad 0 < \|G\| < 1.$$

Then L is slowly varying.

(ii) *If L is slowly varying and* (7.14) *holds, then* $b_n \to b$ *and* $G = \|G\|\delta_b$. *Furthermore, given* $0 < \beta < 1$, *if* (a'_n) *is picked to satisfy* $nL(a'_n) \to -\log\beta$ *(which is always possible by Proposition* 7.4*) then*

$$\mathscr{L}\left(\sum_{i=1}^n X_i/a'_n\right) \to \beta\delta_0.$$

PROOF. By Proposition 7.5 we can find $c_n \nearrow \infty$, $c_{n+1}/c_n \to 1$ such that $\mathscr{L}(c_n^{-1}(S_n - a_n)) \to \|G\|\delta_0$. The boundedness condition on (a_n) also holds for $(c_n a_n)$, and by Proposition 7.3 we conclude that L is slowly varying near infinity. This proves (i).

We now proceed to show (ii). Since $\mathscr{L}(S_n)$ is not dissipative along any subsequence, Corollary 6.5 implies

(7.15) $$\sup_n n \int \frac{x^2}{1+x^2}\, dF(a_n x) < \infty,$$

which shows that

(7.16) $$nL(a_n) \leqq c < \infty, \qquad n \geqq 1.$$

We will now show that (b_n) is a bounded sequence. If not, assume that $b_{n_k} \to \infty$ (the case of $-\infty$ is similar), and write

$$S_n - b_n = \frac{a_n S_n - c_n}{c_n} \cdot b_n, \tag{7.17}$$

where $c_n = a_n b_n$. Since we may assume $c_{n_k} > a_{n_k}$, we have $n_k L(c_{n_k}) \leqq c$ by (7.16). Hence along a subsequence of (n_k) one gets $nL(c_n) \to \lambda \geqq 0$. By Proposition 7.2 along this subsequence $\mathscr{L}(a_n S_n / c_n) \to e^{-\lambda}\delta_0$, and by (7.17), given that $b_{n_k} \to \infty$, we have $\mathscr{L}(S_n - b_n) \to 0$ along this subsequence, which contradicts (7.14). It follows that (b_n) is a bounded sequence. It is now clear from (7.14), (7.16) and Proposition 7.2 that the sequence (b_n) must converge to a real number b. The last part of assertion (ii) follows from Propositions 7.2 and 7.4.

As a corollary we get

THEOREM 7.7. *If $\mathscr{L}(S_n) \to G$, $0 < \| G \| < 1$, then $G = \| G \| \delta_0$.*

PROOF. If G is not concentrated in the origin, then by Proposition 4.1 we have $a_{n+1}/a_n \to 1$. Hence by Theorem 7.6 (with $b_n = 0$) we have L slowly varying. Let β be such that

$$G(\{0\}) < \beta < \| G \|.$$

By Theorem 7.6 there exist $a'_n \nearrow \infty$, such that $\mathscr{L}(\Sigma_{i=1}^n X_i / a'_n) \to \beta\delta_0$. Since the normalizing constants a'_n make more mass go to zero than the constants a_n, we must have $a_n / a'_n \to 0$; on the other hand, a'_n also allow more mass to escape to infinity than a_n, hence $a_n / a'_n \to \infty$, which is a contradiction. It follows that $G = \| G \| \delta_0$.

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